Indices

e is the outer edge of the boundary layer;
w is the body surface;
t is the transition point;
L is the laminar sublayer boundary;
i is the inner sublayer;
0 is the outer sublayer.

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ENTRAINMENT OF A VISCOPLASTIC FLUID BY A MOVING SURFACE

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The thickness of the film remaining on the surface of a vertical plate during its extraction from a viscoplastic liquid is determined theoretically.

One of the most widespread methods of superposing a layer of lubricating fluid on a solid in different technological processes is to extract the solid from the fluid at a constant velocity v_0. Processes to obtain photographic materials, magnetic recorder tapes, cable insulation, etc. are examples.

Let an infinite plate be extracted vertically upward at a constant velocity v_0 from a sufficiently large vessel with a fluid. Far from the plate the fluid is at rest and its surface is horizontal. Let us take this horizontal surface as the origin $x = 0$ and let us direct the y axis perpendicularly to the plate and the x axis upward in the direction of plate motion.

The thickness of the film remaining on the plate surface as it is extracted from the fluid is determined by the interaction between the internal friction forces, the mass forces, and the surface tension force. The degree of influence of each of these forces on the quantity of fluid being entrapped is determined by the physical properties of the fluid, the state of the surface, the velocity of plate extraction, and a number of other factors.

According to the Landa–Levich–Deryagin theory [1, 2], the whole film can be separated into two domains: 1) a zone located sufficiently high above the meniscus and entrained directly by the body (the free boundary of the fluid is almost parallel to the plane of the plate in this domain); 2) the zone of the meniscus, which is deformed somewhat because of the plate motion (the shape of the surface is taken approximately coincident with the static meniscus). The solutions obtained for each domain must then be joined, where the junction condition is continuity of the surface curvature in the domain of small curvatures.

Following this path, we determine the thickness of the film remaining on the plate surface during its extraction from a viscoplastic fluid.

Thus, we have the static meniscus equation for zone 2:

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$$\frac{\frac{d^2h}{dx^2}}{\left[1+\left(\frac{dh}{dx}\right)^2\right]^{3/2}}=\frac{\rho g x}{\sigma}, \tag{1}$$

where h is the film thickness, σ is the coefficient of surface tension, and ρ is the density of the fluid. After a single integration of (1) and substitution of the boundary condition as $x \to 0$

$$\frac{\frac{dh}{dx}}{\left[1+\left(\frac{dh}{dx}\right)^2\right]^{1/2}}=\frac{\rho g x^2}{2\sigma}-1. \tag{2}$$

Hence, from (1) we find the curvature of the static meniscus surface in the juncture domain

$$\left.\frac{d^2h}{dx^2}\right|_{n\to 0}=\left(\frac{2\rho g}{\sigma}\right)^{\frac{1}{2}}. \tag{3}$$

The equation of motion in zone 1 has the form

$$\frac{\partial \tau}{\partial y}-\rho g+\sigma\frac{d^3h}{dx^3}=0 \tag{4}$$

with the boundary conditions

$$u=v_0 \quad \text{for} \quad y=0, \tag{5}$$

$$\tau=0 \quad \text{for} \quad y=h. \tag{6}$$

Let us select the classical Shvedov– Bingham model as the rheological equation of state of a viscoplastic fluid

$$\begin{aligned}&\tau=-\tau_0+\mu\frac{\partial u}{\partial y}, \quad \tau>\tau_0; \quad \frac{\partial u}{\partial y}<0;\\ &\frac{\partial u}{\partial y}=0, \quad \tau\leqslant\tau_0;\end{aligned} \tag{7}$$

where τ is the shear stress, τ_0 is the yield point, and μ is the plastic viscosity.

Integrating (4) with (6) and (7) taken into account, we obtain

$$-\tau_0+\mu\frac{du}{dy}=-\left(\rho g-\delta\frac{d^3h}{dx^3}\right)(h-y). \tag{8}$$

We determine the domain of viscoplastic flow from (8) and the second condition $\partial u/\partial y = 0$ of (7) for $y = \delta$:

$$\tau_0=\left(\rho g-\sigma\frac{d^3h}{dx^3}\right)(h-\delta), \tag{9}$$

i.e.,

$$\delta=h-\Delta, \tag{10}$$

where Δ is the zone of quasisolid flow

$$\Delta=\frac{\tau_0}{\rho g-\sigma\frac{d^3h}{dx^3}}. \tag{11}$$

Equation (8) with the boundary condition (5) has the solution

$$u=v_0+\frac{\tau_0}{\mu}y-\frac{1}{\mu}\left(\rho g-\sigma\frac{d^3h}{dx^3}\right)\left(hy-\frac{y^2}{2}\right). \tag{12}$$

We determine the thickness of the fluid layer entrained by the plate from the continuity equation. In this case it has the form

$$Q=\int_0^\delta u\,dy+\int_\delta^h u\Big|_{y=\delta}dy=\text{const}, \tag{13}$$

which, taking account of (12), (11), and (10) can be represented equivalently as

$$Q=v_0h+\frac{\tau_0h^2}{2\mu}-\frac{h^3}{3\mu}\left(\rho g-\sigma\frac{d^3h}{dx^3}\right)-\frac{\tau_0^3}{6\mu}\left(\rho g-\sigma\frac{d^3h}{dx^3}\right)^{-2}, \tag{14}$$

$$Q=v_0h-\frac{\tau_0}{3\mu}\frac{h^3}{\Delta}\left(1-\frac{3}{2}\frac{\Delta}{h}+\frac{1}{2}\frac{\Delta^3}{h^3}\right), \tag{15}$$

$$Q=v_0h-\frac{\tau_0}{2\mu}\delta^2\frac{\left(1-\dfrac{\delta}{3h}\right)}{\left(1-\dfrac{\delta}{h}\right)}. \tag{16}$$

The general solution of the nonlinear differential equation (14) cannot possibly be obtained. Let us hence consider some particular (degenerate) cases.

1. Let $\delta/h \ll 1$, i.e., the zone of quasisolid flow is much broader than the shear flow. Then we find from (16)

$$Q=v_0h-\frac{\tau_0}{2\mu}\delta^2. \tag{17}$$

The film thickness at a large distance from the fluid surface in the vessel tends to the constant limit h_0 equal to $h = Q/v_0$. Then from (17)

$$\delta=\left[\frac{2\mu v_0}{\tau_0}(h-h_0)\right]^{\frac{1}{2}}. \tag{18}$$

From this and from (10) and (11) there follows

$$\frac{\sigma}{\tau_0}\frac{d^3h}{dx^3}=\frac{\rho g}{\tau_0}-\left\{h-\left[\frac{2\mu v_0}{\tau_0}(h-h_0)\right]^{\frac{1}{2}}\right\}^{-1}. \tag{19}$$

Let us introduce dimensionless variables and parameters

$$z=\frac{x}{h_0},\ \ L=\frac{h}{h_0},\ \text{Ca}=\frac{\mu v_0}{\sigma},\ \ B=\frac{\tau_0}{(\rho g\sigma)^{1/2}},\ D_0=h_0\left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}} \tag{20}$$

into (19). Then

$$\frac{1}{BD_0}\frac{d^3L}{dz^3}=\frac{D_0}{B}-\left\{L\left[1-\left(\frac{2\text{Ca}}{BD_0}\right)^{\frac{1}{2}}\frac{(L-1)^{\frac{1}{2}}}{L}\right]\right\}^{-1}. \tag{21}$$

The boundary conditions can be written in the form

$$L\to 1,\ \frac{dL}{dz}\to 0,\ \frac{d^2L}{dz^2}\to 0 \text{ for } z\to\infty. \tag{22}$$

In the case $(2\text{Ca}/BD_0)^{1/2} \ll 1$, the member $(2\text{Ca}/BD_0)^{1/2}(L-1)^{1/2}/L$ can be neglected in (21) as compared with one, i.e.,

$$\frac{1}{BD_0}\frac{d^3L}{dz^3}=\frac{\frac{D_0}{B}L-1}{L}. \tag{23}$$

Let us seek the linearized solution of the equation obtained.

Let us set $(D_0/B)L = (D_0/B)(1+\varepsilon) = D_0/B$ in the right side, where $\varepsilon(z)$ is a small parameter. Consequently, we arrive at a simplified differential equation

$$\frac{1}{BD_0}\frac{d^3L}{dz^3} = \frac{\frac{D_0}{B} - 1}{L}. \tag{24}$$

which is simplified by the introduction of the new dimensionless coordinate

$$\lambda = -\left[BD_0\left(\frac{D_0}{B} - 1\right)\right]^{\frac{1}{3}} z$$

to the equation:

$$\frac{d^3L}{d\lambda^3} = -\frac{1}{L}. \tag{25}$$

It has been shown in [1] that

$$\left.\frac{d^2L}{d\lambda^2}\right|_{L\to\infty} = 2.35. \tag{26}$$

follows from (25) under the boundary conditions (22).

Since equality of the static and dynamic curvatures of the meniscus is the condition for juncture of the solutions, we obtain from (3) and (26)

$$2.35\,(D_0^2 - BD_0)^{-\frac{2}{3}} = \sqrt{2}\,D_0. \tag{27}$$

Solving the algebraic equation (27) for D_0, we find

$$D_0 = B - 0.109\,[(1 + 18.32\,B)^{-\frac{1}{2}} - 1]. \tag{28}$$

Therefore, there is a domain of extraction velocities $Ca \ll BD_0$ for a viscoplastic fluid where the layer thickness is independent of the extraction velocity, as should hold in a viscous fluid. In this case this quantity is determined by the surface tension coefficient σ, the yield point τ_0, the density ρ, and the acceleration of gravity g.

2. Let $[1 - (3/2)(\Delta/h)] \gg (1/2)(\Delta^3/h^3)$. We then have from (15)

$$Q = v_0 h - \frac{\tau_0}{3\mu}\frac{h^3}{\Delta}\left(1 - \frac{3}{2}\frac{\Delta}{h}\right). \tag{29}$$

Let us assume that

$$1 - \frac{3}{2}\frac{\Delta}{h} \geqslant 10\frac{\Delta^3}{2h^3}, \text{ hence } \frac{\Delta}{h} < \frac{1}{2}. \tag{30}$$

Therefore, (29) is valid for the sufficiently general case for which the domain of quasisolid flow can be of the same order as the viscoplastic flow zone.

After manipulating (29) with (10) taken into account, we obtain

$$\frac{\sigma}{3\mu}h^3\frac{d^3h}{dx^3} - \frac{\rho g h^3}{3\mu} + \frac{\tau_0 h^2}{2\mu} + v_0 h = Q. \tag{31}$$

The layer thickness h, sufficiently high above the meniscus, tends asymptotically (as $x \to \infty$) to the constant value h_0. All the derivatives of h with respect to x hence tend to zero. Substituting the appropriate quantities into (31), we find

$$Q = v_0 h_0 + \frac{\tau_0}{2\mu}h_0^2 - \frac{\rho g}{3\mu}h_0^3. \tag{32}$$

Let us combine (31) and (32). By introducing the dimensionless variables and parameters:

$$z=\frac{x}{h_0},\ L=\frac{h}{h_0},\ D_0=h_0\left(\frac{\rho g}{\sigma}\right)^{\frac{1}{2}},\ \mathrm{Ca}=\frac{\mu v_0}{\sigma},\ B=\frac{\tau_0}{(\rho g\sigma)^{1/2}} \tag{33}$$

we arrive at the differential equation

$$\frac{L^3}{3\mathrm{Ca}}\frac{d^3L}{dz^3}=(1-L)+\frac{BD_0}{2\mathrm{Ca}}(1-L^2)-\frac{D_0^2}{3\mathrm{Ca}}(1-L^3) \tag{34}$$

with the boundary conditions

$$L\to 1,\ \frac{dL}{dz}\to 0,\ \frac{d^2L}{dz^2}\to 0 \text{ for } z\to\infty. \tag{35}$$

Let us seek the linearized solution of the problem (34) and (35). Let us set $L = 1 + \varepsilon$, where $\varepsilon(z)$ is a small parameter, and we find by neglecting terms greater than the quadratic in ε

$$\frac{L^3}{3\mathrm{Ca}}\frac{d^3L}{dz^3}=(1-L)\left(1+\frac{BD_0}{\mathrm{Ca}}-\frac{D_0^2}{\mathrm{Ca}}\right). \tag{36}$$

Let us introduce the new variable

$$\lambda=\left[3\mathrm{Ca}\left(1+\frac{BD_0}{\mathrm{Ca}}-\frac{D_0^2}{\mathrm{Ca}}\right)\right]^{-\frac{1}{3}}z. \tag{37}$$

Then

$$\frac{d^3L}{d\lambda^3}=\frac{1-L}{L^3}. \tag{38}$$

It has been shown in [1, 2] that

$$\left.\frac{d^2L}{d\lambda^2}\right|_{L\to\infty}=0.64 \tag{39}$$

follows from the differential equation (38) with the boundary conditions (35). The juncture condition, i.e., the equality of (3) and (39), yields

$$0.64\left[3\mathrm{Ca}\left(1+\frac{BD_0}{\mathrm{Ca}}-\frac{D_0^2}{\mathrm{Ca}}\right)\right]^{-\frac{2}{3}}=\sqrt{2}\,D_0$$

or

$$D_0=0.94(\mathrm{Ca}+BD_0-D_0^2)^{\frac{2}{3}}. \tag{40}$$

As follows from (3), formula (40) is valid in the case $2B < D_0$, and determines the layer thickness h_0 for known values of the rheological parameters τ_0 and μ, the velocity of extraction v_0, the surface tension coefficient σ, and given values of ρ and g.

For $\mathrm{Ca} \gg BD_0$ we find from (40) that

$$D_0=0.94(\mathrm{Ca}-D_0^2)^{\frac{2}{3}}, \tag{41}$$

i.e., the film thickness is independent of the plastic properties of the fluid.

Thus, the layer thickness of a viscoplastic fluid is independent of Ca for extraction velocities $\mathrm{Ca} \ll BD_0$ and is determined from (27), which yields $D_0 = f_1(B)$. When $\mathrm{Ca} \gg BD_0$, the film thickness is independent of B and is calculated from the relationship (41), where $D_0 = f_2(\mathrm{Ca})$.

In the general case of an arbitrary velocity of plate extraction, it is possible to write

$$\bar{D}_0=f(B,\mathrm{Ca}). \tag{42}$$

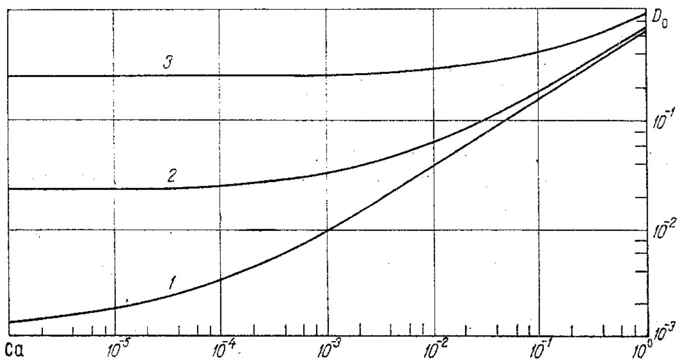

Fig. 1. Dependence of the film thickness D_0 on the extraction velocity Ca for the plasticity parameter B: 1) 0.02; 2) 0.1; 3) 0.5.

Let us consider the rheological equation of state (7). It contains the equations of state of a viscous fluid ($\tau_0 = 0$) and of a plastic body ($\mu = 0$) as limit cases, and combines them additively. The relations (28) and (41) yield a solution of the problem in these two limit cases. Hence, let us represent the general interpolation formula in the form of a sum of the two limit cases (28) and (41) according to the form of the rheological law (7):

$$\bar{D}_0 = f_1(B) + f_2(\mathrm{Ca}). \tag{43}$$

Certain results of calculations using (43) are represented in Fig. 1. Formula (43) goes over into the limit cases $\mathrm{Ca} \ll BD_0$ and $\mathrm{Ca} \gg BD_0$, respectively, in (28) and (41), which are given a theoretical foundation. Moreover, the results determined by means of (43) and (40) agree quite well in the domain where the solution (40) is valid. It can therefore be hoped that (43) possesses sufficient accuracy and correctly transfers the dependence of $\bar{D}_0$ on Ca and B in the whole range of numbers Ca.

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